



TITLE:

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CITATION:

OKA, Mutsuo. ON THE DEFORMATION OF A CERTAIN TYPE OF ALGEBRAIC VARIETIES. 数理解析研究所講究録 1987, 634: 72-92

ISSUE DATE:

1987-12

URL:

<http://hdl.handle.net/2433/100089>

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ON THE DEFORMATION OF A CERTAIN TYPE OF ALGEBRAIC VARIETIES

Mutsuo OKA

Dedicated to Professor I. Tamura for his 60th birthday

§1. Introduction

Let $A = (a_{ij})$ ($1 \leq i, j \leq n$) be an upper triangular integral matrix with a non-zero determinant and $a_{ij} \geq 0$ for each i, j . Let Δ be the n -simplex in \mathbb{R}^n which is spun by $A_0 = \vec{0}$ and $A_i = (a_{i1}, \dots, a_{in})$ ($i=1, \dots, n$). Let $A_{n+1}, A_{n+2}, \dots, A_\ell$ be the other integral points in Δ . For an integral vector $\nu = (\nu_1, \dots, \nu_n)$, we denote the monomial $y_1^{\nu_1} \dots y_n^{\nu_n}$ by y^ν . For $t = (t_0, \dots, t_\ell)$ of $\mathbb{C}^{\ell+1}$, we define

$$(1.1) \quad h(y, t) = t_0 + \sum_{j=1}^{\ell} t_j y^{A_j}$$

and let M_t^a be the affine variety in \mathbb{C}^n defined by $h(y, t) = 0$. There exists a toric variety W of dimension n which depends only on Δ and a Zariski open subset U of $\mathbb{C}^{\ell+1}$ such that $W \supset \mathbb{C}^n \supset M_t^a$ and the closure M_t of M_t^a in W is non-singular for each $t \in U$. This type of algebraic variety M_t appears as an exceptional divisor of a resolution of an

isolated hypersurface singularity ([12]). The purpose of this paper is to study this deformation $\{M_t\}$ in W .

In §5, we prove the surjectivity of the infinitesimal displacement map

$$\xi : T_t U \rightarrow H^0(M_t, \nu_t).$$

In §6, we give a criterion about the injectivity of the Kodaira-Spencer map

$$\delta \cdot \xi^e : T_t U^e \longrightarrow H^1(M_t, \theta_t).$$

In §7, we will apply the results in §§5,6 to construct a complete deformation of a Godeaux surface.

§2. Infinitesimal displacement

Let W be a compact complex manifold of dimension n and let $\{M_t\}$ ($t \in U$) be an analytic family of non-singular hypersurfaces where U is an open set of \mathbb{C}^{l+1} . Let $\{(U_\alpha, z_\alpha)\}$ ($\alpha \in S$) be local coordinate systems of W such that (i) $W = \bigcup_{\alpha \in S} U_\alpha$ and (ii) there exists analytic functions $f_\alpha(z_\alpha, t)$ on $U_\alpha \times U$ such that $M_t \cap U_\alpha = \{z_\alpha \in U_\alpha ; f_\alpha(z_\alpha, t) = 0\}$. Let $h_{\alpha\beta} = f_\alpha / f_\beta$. We may assume that $h_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. The line bundle $[M_t]$ is defined by the cocycle $\{h_{\alpha\beta}\}$ of $H^1(W, \mathcal{O}^*)$ and the normal bundle N_t of M_t in W is the restriction of $[M_t]$ to M_t . Let ν_t be the sheaf of the germs of the holomorphic sections of N_t . Take a holomorphic tangent vector $v \in T_t U$. As $f_\alpha = h_{\alpha\beta} f_\beta$, we have

$$(2.1) \quad v(f_\alpha) = h_{\alpha\beta} v(f_\beta) \quad \text{on } U_\alpha \cap U_\beta \cap M_t.$$

This defines a canonical linear mapping

$$(2.2) \quad \xi : T_t U \rightarrow H^0(M_t, \nu_t)$$

where $\xi(v) = \{v(f_\alpha)\}(\alpha \in S)$. $\xi(v)$ is called the infinitesimal displacement along v .

Let θ_W and θ_t be the sheaves of the germs of holomorphic vector fields of W and M_t respectively. We have the exact sequence of sheaves:

$$(2.3) \quad 0 \rightarrow \theta_t \rightarrow \theta_W|_{M_t} \rightarrow \nu_t \rightarrow 0.$$

This induces the following exact sequence.

$$(2.4) \quad 0 \rightarrow H^0(M_t, \theta_t) \rightarrow H^0(M_t, \theta_W|_{M_t}) \rightarrow H^0(M_t, \nu_t) \\ \xrightarrow{\delta} H^1(M_t, \theta_t) \longrightarrow H^1(M_t, \theta_W|_{M_t}) \longrightarrow \dots$$

The composition

$$(2.5) \quad T_t U \xrightarrow{\xi} H^0(M_t, \nu_t) \xrightarrow{\delta} H^1(M_t, \theta_t)$$

is equal to the infinitesimal deformation map. See Kodaira-Spencer [6] or Kodaira [7] for details.

§3. Resolution of a hypersurface singularity

We recall basic properties about the resolution of a hypersurface singularity through the toroidal embedding theory. We use the same notation as in [12]. Let $f(z_0, \dots, z_n) = \sum_{\nu} a_{\nu} z^{\nu}$ be an analytic function defined in a

neighborhood of the origin and we assume that $V = f^{-1}(0)$ has an isolated singular point at the origin. Let $\Gamma_+(f)$ be the convex hull of $\bigcup_{a_\nu \neq 0} \{\nu + (\mathbb{R}^+)^{n+1}\}$. The Newton boundary $\Gamma(f)$

is the union of the compact faces of $\Gamma_+(f)$. We assume that f is non-degenerate on each face Δ of $\Gamma(f)$. Let N be the dual space $\text{Hom}(\mathbb{R}^{n+1}, \mathbb{R})$. We identify N with \mathbb{R}^{n+1} through the standard inner product and we denote the dual vectors by column vectors to avoid confusion. Let N^+ be the set of non-negative dual vectors. We introduce an equivalence relation \sim in N^+ by $P \sim Q$ if and only if $\Delta(P) = \Delta(Q)$. Here $\Delta(P)$ is the locus where the restriction of P on $\Gamma_+(f)$ takes its minimal value which we denote by $d(P)$. This induces a cone-like polyhedral decomposition of N^+ and we denote this by $\Gamma^*(f)$. Let Σ^* be a unimodular simplicial subdivision. For each n -simplex $\sigma = (P_0, \dots, P_n) = (p_{ij})$ which is a unimodular matrix, we associate an affine space C_σ^{n+1} with coordinate $y_\sigma = (y_{\sigma 0}, \dots, y_{\sigma n})$. Let $\pi_\sigma : C_\sigma^{n+1} \rightarrow \mathbb{C}^{n+1}$ be the birational morphism defined by $\pi(y_\sigma) = (z_0, \dots, z_n)$ where $z_i = \prod_{j=0}^n y_{\sigma j}^{p_{ij}}$. Let X be the complex manifold of dimension $n+1$ which is obtained by gluing the affine spaces C_σ^{n+1} where σ moves in the n -simplices of Σ^* and let $\hat{\pi} : X \rightarrow \mathbb{C}^{n+1}$ be the projection map. Let \tilde{V} be the proper transform of V and let $\pi : \tilde{V} \rightarrow V$ be the restriction of $\hat{\pi}$ to \tilde{V} . By the non-degeneracy assumption, $\pi : \tilde{V} \rightarrow V$ is a good resolution of V . For each strictly positive vertex P of Σ^* with $\dim \Delta(P) \geq 1$, there are corresponding exceptional divisors $\hat{E}(P)$ and $E(P)$

of $\hat{\pi}$ and π respectively so that $E(P)$ is a hypersurface in $\hat{E}(P)$. $\hat{E}(P)$ is a toric variety. Let $\sigma = (P_0, \dots, P_n)$ with $P = P_0$. Then in the coordinate chart C_σ^{n+1} , $\hat{E}(P)$ is defined by $y_{\sigma 0} = 0$ and $E(P)$ is defined by $\hat{E}(P) \cap \{h_\sigma(y_{\sigma 1}, \dots, y_{\sigma n}) = 0\}$ where $h_\sigma(y_\sigma)$ is defined by

$$(3.1) \quad f_{\Delta(P)}(\pi_\sigma(y_\sigma)) = \prod_{i=1}^n y_{\sigma i}^{d(P_i)} h_\sigma(y_{\sigma 1}, \dots, y_{\sigma n}).$$

§4. Compactification of M_t^a .

Let $h(y, t)$ be as in (1.1). Let σ' be the unimodular matrix (P, R_1, \dots, R_n) where $P = {}^t(1, \dots, 1)$, $R_1 = {}^t(0, 1, \dots, 0)$. Let $\pi_{\sigma'} : C^{n+1} \rightarrow C^{n+1}$ be as in §3. Let y_0, \dots, y_n be the coordinate of the source. Then we have $z_0 = y_0$ and $z_i = y_0 y_i$ for $i = 1, \dots, n$. Let k be the degree of h and we define $f_\Xi(z, t) = h(\pi_{\sigma'}^{-1}(z, t)) z_0^k = h(z_1/z_0, \dots, z_n/z_0, t) z_0^k$. Then $f_\Xi(z, t)$ is a homogeneous polynomial in z_0, \dots, z_n and we can write

$$(4.1) \quad f_\Xi(z, t) = \sum_{j=0}^l t_j z^B$$

for some integral vectors B_0, \dots, B_l . Note that $B_0 = (k, 0, \dots, 0)$. Let $f(z, t) = f_\Xi(z, t) + \sum_{i=0}^n z_i^L$ for a sufficiently large L . The notation $f_\Xi(z)$ is the same as in [12] if we set $\Xi = \Delta(P)$. There exists a Zariski open subset U of C^{l+1} such that $f(z, t)$ has a non-degenerate Newton boundary for each $t \in U$. Let $\sigma = (P, P+R_1, \dots, P+R_n)$. If L is

sufficiently large, $\Delta(P+P_i) \supset B_0$ for each $i = 1, \dots, n$. Thus σ is an admissible simplex of $\Gamma^*(f)$. (σ' is not necessarily an admissible simplex.) Thus we can take a unimodular simplicial subdivision Σ^* which has σ as an n -simplex by §3 of [12].

Assertion. The defining equation of $E(P)$ in $C_\sigma^{n+1} \cap \{y_{\sigma 0} = 0\}$ is equal to $h(y_\sigma, t) = 0$.

Proof. $E(P)$ is defined by $h_\sigma(y_\sigma, t) = 0$ where

$$\begin{aligned} h_\sigma(y_\sigma, t) &= f_\Delta(\pi_\sigma(y_\sigma), t) / y_{\sigma 0}^{d(P)} \prod_{i=1}^n y_{\sigma i}^{d(P)+d(R_i)} \\ &= f_\Delta(\pi_\sigma(y)) / \{(y_{\sigma 0} \dots y_{\sigma n})^{d(P)} \prod_{i=1}^n y_{\sigma i}^{d(R_i)}\} \\ &= h(y, t) = h(y_\sigma, t) \end{aligned}$$

Here we have used the equality $\pi_{\sigma'}^{-1} \cdot \pi_\sigma = \pi_{\sigma', -1_\sigma}$ and $y_0 = y_{\sigma 0} \dots y_{\sigma n}$ and $y_i = y_{\sigma i}$ for $i = 1, \dots, n$.

Thus we take $E(P)$ as the compactification M_t of M_t^a and $\hat{E}(P)$ as W hereafter. Note that $\pi_1(M_t)$ is a finite cyclic group by Theorem (7.3) of [12]. Let S be the set of the n -simplex τ of Σ^* such that P is a vertex of τ . Then it is obvious that $\{C_\sigma^n\} (\sigma \in S)$ is an open covering of W where $C_\sigma^n = C_\sigma^{n+1} \cap \{y_{\sigma 0} = 0\}$.

Remark. To study the deformation of M_t in W , we only need the information about S .

§5. Main theorem

We are ready to state the main theorem. Let ν_t be the sheaf of the germs of the holomorphic sections of the normal bundle N_t of M_t in W . Let q be as in §1.

Theorem (5.1). (i) $\dim H^0(M_t, \nu_t) = q$ and the infinitesimal displacement map $\xi : T_t U \rightarrow H^0(M_t, \nu_t)$ is surjective. The kernel of ξ is generated by $\sum_{j=0}^q t_j \frac{\partial}{\partial t_j}$.

(ii) Let ψ_1, \dots, ψ_q be a system of the generators of $H^0(M_t, \nu_t)$ and let $\Psi : M_t \rightarrow P^{q-1}$ be the associated mapping. Then Ψ is a birational morphism.

Let $W = \hat{E}(P)$ and $M_t = E(P)$ as in §4. For each n -simplex $\tau = (Q_0(\tau), \dots, Q_n(\tau))$ of S , we may assume that

$$(5.2) \quad Q_0(\tau) = P.$$

Let $h_\tau(y_\tau, t)$ be the defining polynomial of M_t in $C_\tau^n = C_\tau^{n+1} \cap \{y_{\tau 0} = 0\}$. h_τ is defined by the equality

$$(5.3) \quad f_\Delta(\pi_\tau(y_\tau), t) = \prod_{i=0}^n y_{\tau i}^{d(Q_i(\tau))} h_\tau(y_\tau, t).$$

Take two simplices α and β in S and let $\alpha^{-1}\beta = (\lambda_{ij})$ ($0 \leq i, j \leq n$). By (5.2), we have $\lambda_{00} = 1$ and $\lambda_{i0} = 0$ for $i = 1, \dots, n$. Recall that C_α^n and C_β^n are glued by

$$(5.4) \quad y_{\alpha i} = \prod_{j=1}^n y_{\beta j}^{\lambda_{ij}} \quad (i = 1, \dots, n).$$

Now we consider the line bundle $[M_t]$ which is defined by the cocycle $\{h_{\alpha\beta}\}$ where $h_{\alpha\beta} = h_\alpha / h_\beta$. By (5.3), we have

$$(5.5) \quad h_{\alpha\beta}(y_\beta, t) = \frac{\prod_{i=0}^n y_{\beta i}^{d(Q_i(\beta))}}{\prod_{i=0}^n y_{\alpha i}^{d(Q_i(\alpha))}}.$$

Here the right hand is considered as a monomial of $y_{\beta 1}, \dots, y_{\beta n}$ through (5.4). The exponent of $y_{\beta 0}$ is zero. We can write $h_\tau(y_\tau, t)$ more explicitly as

$$(5.6) \quad h_\tau(y_\tau, t) = \sum_{j=0}^l t_j y_\tau^{A_j(\tau)}$$

where the positive integral vector $A_j(\tau)$ is characterized by

$$(5.7) \quad \pi_\tau(y_\tau)^{B_j} = \left(\prod_{i=0}^n y_{\tau i}^{d(Q_i(\tau))} \right) y_\tau^{A_j(\tau)}.$$

Combining (5.7) and (5.5), we obtain

$$(5.8) \quad y_\alpha^{A_j(\alpha)} = h_{\alpha\beta} y_\beta^{A_j(\beta)}.$$

(5.8) says that $\{y_\alpha^{A_j(\alpha)}\}_{(\alpha \in S)}$ is an element of $H^0(W, \mathcal{O}([M_t]))$. Thus we get the inequality $\dim H^0(W, \mathcal{O}([M_t])) \geq l + 1$. On the other hand, take a monomial y_σ^μ where $\mu \neq A_j(\sigma)$ for $j = 0, \dots, l$. (Here σ is fixed.) Let Π_k be the hyperplane which contains $\{A_i(\sigma); i \neq k, 0 \leq i \leq n\}$. Then there is an integer k ($0 \leq k \leq n$) such that $A_k(\sigma)$ and μ are separated by Π_k . Take a simplex $\beta = (P, Q_1(\beta), \dots, Q_n(\beta))$ such that

$$(5.9) \quad B_i \in \Delta(Q_1(\beta)) \text{ for } i \neq k, \quad i = 0, \dots, n.$$

Assume that $y_\sigma^\mu = h_{\sigma\beta} y_\beta^\nu$ for $\nu = (\nu_1, \dots, \nu_n)$. Then by the assumption, we have $\nu_1 < 0$. This implies that the section y_σ^μ of $H^0(\mathbb{C}^n, ([M_t]))$ cannot be holomorphically extended to W . Thus using GAGA-principle [13], we have proved the following.

Lemma (5.10). $\dim H^0(W, \mathcal{O}([M_t])) = \ell + 1$ and $\{y_\alpha^{A_j(\alpha)}\}_{(\alpha \in S), (j = 0, \dots, \ell)}$ gives a canonical basis.

This is a special case of §6 of [1] and Lemma 2.3 of [10]. For the further geometry of the toric variety W , see [5, 2, 1, 9, 3].

We are ready to prove (i) of Theorem (5.1). From the exact sequence of sheaves on W :

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}([M_t]) \longrightarrow \nu_t \longrightarrow 0,$$

we have the exact sequence

$$(5.11) \quad 0 \rightarrow \mathbb{C} \longrightarrow H^0(W, \mathcal{O}([M_t])) \xrightarrow{\theta} H^0(M_t, \nu_t) \rightarrow 0.$$

Here we have used the fact that $H^1(W,) = 0$ because W is simply connected ([1]). Thus $\dim H^0(M_t, \nu_t) = \ell$ and $H^0(M_t, \nu_t)$ is generated by $\varphi_j = \{y_\alpha^{A_j(\alpha)}\}_{\alpha \in S}$ ($j = 0, \dots, \ell$). They satisfy the obvious relation $\sum_{j=0}^{\ell} t_j \varphi_j = 0$. Now we study the infinitesimal displacement map $\xi : T_t U \rightarrow H^0(M_t, \nu_t)$. By the definition of ξ , we have

$$\xi\left(\frac{\partial}{\partial t_j}\right) = \left\{\frac{\partial h_\alpha}{\partial t_j}\right\}_{\alpha \in S} = \{y_\alpha^{A_j(\alpha)}\}_{\alpha \in S} = \varphi_j.$$

Thus ξ is surjective and the kernel of ξ is generated by $\sum_{j=0}^l t_j \frac{\partial}{\partial t_j}$. This completes the proof of (i) of Theorem (5.1).

Now we will prove (ii) of Theorem (5.1). Let $\varphi_0, \dots, \varphi_l$ be as above and define $\hat{\varphi} : W \rightarrow P^l$ by $\hat{\varphi}(x) = [\varphi_0(x); \dots; \varphi_l(x)]$. Let $\tau \in S$. As the polynomial $h_\tau(y_\tau)$ contains a non-zero constant term, there exists an integer $0 \leq k \leq n$ such that $A_k(\tau) = (0, \dots, 0)$. As $\hat{\varphi}(y_\tau) = [y_\tau^{A_0(\tau)}; \dots; y_\tau^{A_l(\tau)}]$ on C_τ^n , this implies that $\hat{\varphi}$ is a morphism. We have to prove that $\hat{\varphi}$ is generically injective. Note that $\{A_0(\tau), \dots, A_l(\tau)\}$ is equal to the set of the integral points of the simplex spun by $A_j(\tau)$ ($j = 0, \dots, n$). By Lemma (3.8) of [12], there exist $0 \leq i_1 < \dots < i_n \leq l$ such that $t_\xi = (t_{A_{i_1}(\tau)}, \dots, t_{A_{i_n}(\tau)})$ is a unimodular matrix. Let $\xi^{-1} = (\xi_{ij})$. The image of $\hat{\varphi}|_{C_\tau^n}$ is in the coordinate chart $U_k = \{X_k \neq 0\}$ of P^l . Let $Y_j = X_j / X_k$ ($j \neq k$). Assume that $\hat{\varphi}(y_\tau) = (Y_j)_{j \neq k}$ for $y_\tau \in (C_\tau^*)^n$. Then y_τ is determined by $y_{\tau m} = \prod_{j=1}^n Y_{i_j}^{\xi_{mj}}$ ($m = 1, \dots, n$). This proves that $\hat{\varphi}$ is injective on $(C_\tau^*)^n$. Therefore the restriction of $\hat{\varphi}$ to M_t is also a morphism and is injective on $M_t \cap (C_\tau^*)^n$. The image of $\hat{\varphi}|_{M_t}$ is in the hyperplane $H: \sum_{j=0}^l t_j X_j = 0$ of P^l . Identifying H with P^{l-1} , we have $\hat{\varphi}|_{M_t} = \psi$. This completes the proof of Theorem (5.1).

Remark. If $A = dI_n$, W is the projective space of

dimension n and $\{M_t\}$ are projective hypersurfaces of degree d . This case is studied in [6].

§6. Canonical vector fields

Let $\tau \in S$. Then $\theta_W|C_\tau^n$ is a free \mathbb{C} -module of rank n with a canonical basis $\{\frac{\partial}{\partial y_{\tau 1}}, \dots, \frac{\partial}{\partial y_{\tau n}}\}$. We define $\frac{\tilde{\partial}}{\partial y_{\tau i}} = y_{\tau i} \frac{\partial}{\partial y_{\tau i}}$ for $i=1, \dots, n$. Similarly we define $\tilde{dy}_{\tau i} = \frac{dy_{\tau i}}{y_{\tau i}}$. Let $\beta \in S$ and let $\beta^{-1}\tau = (\lambda_{ij})$ and let $(\mu_{ij}) = \tau^{-1}\beta$. Then we have

Proposition (6.1). (i) We have the formula

$$\frac{\tilde{\partial}}{\partial y_{\tau i}} = \sum_{j=1}^n \lambda_{ji} \frac{\tilde{\partial}}{\partial y_{\beta j}}, \quad \tilde{dy}_{\tau i} = \sum_{j=1}^n \mu_{ij} \tilde{dy}_{\beta j}.$$

(ii) $\{\frac{\tilde{\partial}}{\partial y_{\tau i}} ; i = 1, \dots, n\}$ can be holomorphically extended to W .

Proof. Recall that $y_{\beta j} = \prod_{i=1}^n y_{\tau i}^{\lambda_{ji}}$. Thus the assertion (i) is obvious. The assertion (ii) follows from (i).

Definition (6.2). $\{\frac{\tilde{\partial}}{\partial y_{\tau 1}}, \dots, \frac{\tilde{\partial}}{\partial y_{\tau n}}\}$ generates a subspace of dimension n of $H^0(W, \theta_W)$ which we denote by $\text{Can}(W, \theta_W)$. The restriction of $\text{Can}(W, \theta_W)$ to $H^0(M_t, \theta_W|_{M_t})$ is denoted by $\text{Can}(M_t, \theta_W)$. We call vector fields in $\text{Can}(W, \theta_W)$ or in $\text{Can}(M_t, \theta_W)$ canonical vector fields. These vector fields come from the torus action on W . It is easy to see that

$$\dim \text{Can}(M_t, \theta_W) = n.$$

Corollary (6.3). We have the inequalities
 $\dim H^0(W, \theta_W) \geq n$ and $\dim H^0(M_t, \theta_W | M_t) \geq n$.

Now we characterize the image of
 $\theta : \text{Can}(M_t, \theta_W) \rightarrow H^0(M_t, \nu_t)$. Let σ be the fixed simplex so
 that $h_\sigma(y_\sigma, t) = h(y_\sigma, t)$ where h is as in (1.1). Let X
 $\in H^0(M_t, \theta_W | M_t)$ and let $X = \sum_{i=1}^n X_{\tau i} \frac{\partial}{\partial y_{\tau i}}$ on C_τ^n . Then it is
 easy to see that

$$(6.4) \quad \theta(X) = (\theta(X)_\tau)_{\tau \in S} \quad \text{where} \quad \theta(X)_\tau = \sum_{i=1}^n X_{\tau i} \frac{\partial h_\tau}{\partial y_{\tau i}}.$$

Let X^1, \dots, X^n be the canonical vector fields defined by

$$(6.5) \quad X^i = \frac{\tilde{\partial}}{\partial y_{\sigma i}} = y_{\sigma i} \frac{\partial}{\partial y_{\sigma i}} \quad \text{on } C_\sigma^n \quad (i = 1, \dots, n).$$

Then we have

$$(6.6) \quad \theta(X^i)_\sigma = y_{\sigma i} \frac{\partial h}{\partial y_{\sigma i}} \quad (i = 1, \dots, n).$$

We claim that $\{\theta(X^i)\}$ ($i = 1, \dots, n$) are linearly independent. In fact, assume that $\sum_{i=1}^n \lambda_i \theta(X^i) = 0$. Then we must

$$\text{have } \sum_{j=1}^n t_j b_j y_\sigma^j \equiv 0 \quad \text{modulo } h(y_\sigma, t) \quad \text{where } b_j = \sum_{i=1}^n \lambda_i a_{ji}.$$

This implies that $\lambda_i = 0$ for each i . Thus we have shown

Theorem (6.7). $\theta(X^1), \dots, \theta(X^n)$ are linearly independent. They are characterized by

$$\theta(X^i)_\sigma = \frac{d}{ds} h(y_{\sigma 1}, \dots, s y_{\sigma i}, \dots, y_{\sigma n}, t) |_{s=1}.$$

Now we consider the following subfamily of $\{M_t\}$. Let $U^e = \{t \in U ; t_0 = \dots = t_n = 1\}$. We call $\{M_t\} (t \in U^e)$ the embedded deformation. Let $\xi^e : T_t U^e \rightarrow H^0(M_t, \nu_t)$ be the restriction of ξ to $T_t U^e$. Then we have

Theorem (6.8). Assume that $H^0(M_t, \theta_W|_{M_t}) = \text{Can}(M_t, \theta_W)$. Then the Kodaira-Spencer map $\delta \cdot \xi^e : T_t U^e \rightarrow H^1(M_t, \theta_t)$ is injective and $H^0(M_t, \theta_t) = 0$.

Proof The second assertion is immediate from Theorem (6.7), (2.4) and the assumption. Assume that $\delta \cdot \xi^e(v) = 0$ where $v = \sum_{j=n+1}^l \lambda_j \frac{\partial}{\partial t_j}$. Then by (2.4), we can write $(\xi^e(v))_\sigma = \sum_{i=1}^n \mu_i y_{\sigma i} \frac{\partial h}{\partial y_{\sigma i}}$ for some complex μ_1, \dots, μ_n . This implies that

$$\sum_{k=1}^n \left(\sum_{i=1}^n \mu_i a_{ki} \right) y_\sigma^{A_k} + \sum_{k=n+1}^l (\lambda_k + \sum_{i=1}^n \mu_i a_{ki}) y_\sigma^{A_k} = 0$$

modulo $h(y_\sigma, t)$. This implies that $\lambda_k = 0$ for $k = n+1, \dots, l$ and $\mu_i = 0$ for $i=1, \dots, n$, because the left side has no constant term. This completes the proof. It seems that the assumption in Theorem (6.8) is satisfied in many cases if W is not projective space P^n . The following is an example where the Kodaira-Spencer map is not injective.

Example (6.9). (Hashimoto- Oka[4]) Let M be the algebraic surface which is the compactification of $y_1 + y_1^9 y_2^{16} + y_1^3 y_3^4 + 1 = 0$. Then M has the following invariants: $K^2 = 0$, $p_g = 1$ and $\pi_1(M) = \mathbb{Z}/2\mathbb{Z}$. M has 27 dimensional

effective deformation and $\dim H^1(M, \theta_W|_M) = 20$. On the other hand, $H^0(M, \theta_W|_M) = 12$ and the dimension of the image of effective deformation is 18.

§7. Deformation of a Godeaux surface.

In this section, we study the case of $n = 3$. Recall that $E = \Delta(P)$ is spun by B_0, \dots, B_3 . Let E_i be the 2-face of E with $B_i \notin E_i$ for $i=0, \dots, 3$. Let P_0, \dots, P_3 be the vertices of Σ^* which are adjacent to P such that $\Delta(P_i) \supset E_i$. We define divisors \hat{C}_i of W by $\hat{E}(P) \cap \hat{E}(P_i)$ and divisors C_i of M by $E(P) \cap E(P_i)$ for $i=0, \dots, 3$. Let σ be as in §4 and we denote $y_{\sigma i}$ by y_i for simplicity. Let $A = \mathbb{C}[y_1, y_1^{-1}, \dots, y_3, y_3^{-1}]$. For a polynomial $g(y)$ of A , we define an integer $\text{ord}_{\hat{C}_i} g(y)$ by the order of the zeros (or poles) of $g(y)$ along the divisor \hat{C}_i . Similarly we define $\text{ord}_{C_i} g(y)$ by the order of the zeros of $(g|_{M_t})$ along C_i . In general, we have the inequality $\text{ord}_{\hat{C}_i} g \leq \text{ord}_{C_i} g$.

Definition (7.1). We say that $g(y)$ has a regular form on C_i if $\text{ord}_{\hat{C}_i} g(y) = \text{ord}_{C_i} g(y)$.

We fix an index a for $0 \leq a \leq 3$. Let $\tau = (P, Q_1(\tau), Q_2(\tau), Q_3(\tau))$ be a simplex of S such that $Q_1(\tau) = P_a$ and let $\sigma^{-1} \cdot \tau = (\lambda_{ij})$. Then by the definition, we have $\text{ord}_{\hat{C}_a} y^\nu = \sum_{j=1}^3 \nu_j \lambda_{j1}$. We define $h^a(y, t) = \sum' t_j y^{A_j}$

where the sum is taken for j such that $B_j \in \Xi_a$. Note that $h^a(y, t)$ is homogeneous with respect to the weight $(\lambda_{11}, \lambda_{21}, \lambda_{31})$ and

$$(7.2) \quad \text{ord}_{\hat{C}_a} y^A h^a > \text{ord}_{\hat{C}_a} h^a.$$

Note also that h^a is irreducible in A , because C_a is an irreducible curve and the defining polynomial of C_a is h^a up to the multiplication of a monomial. Take $g \in A$. Let $k = \text{ord}_{\hat{C}_i} g$ and let g_k be the leading term of g with respect to the above weight. Then we have

Lemma (7.3). g has a regular form on C_a if and only if g_k is not zero modulo h^a .

Proof. We can write $g_k(y(y_\tau)) = y_{\tau 1}^k g'(y_{\tau 2}, y_{\tau 3})$. As $g(y(y_\tau)) \equiv g_k(y(y_\tau))$ modulo $(y_{\tau 1}^{k+1})$, it is easy to see that $g'|_{C_a} \equiv 0$ iff $g_k \equiv 0$ modulo h^a .

Now let $X = \sum_{j=1}^3 X_j \frac{\tilde{\partial}}{\partial y_j}$ be a rational vector field on W such that $X_j \in A$. We define $\text{ord}_{\hat{C}_i} X = \text{minimum}_{1 \leq j \leq 3} \text{ord}_{\hat{C}_i} X_j$ and $\text{ord}_{C_i} X = \text{minimum}_{1 \leq j \leq 3} \text{ord}_{C_i} X_j$. Let $X = \sum_{j=1}^3 X_{\tau j} \frac{\tilde{\partial}}{\partial y_{\tau j}}$ on C_τ^3 . Then we have $\text{minimum}_{1 \leq j \leq 3} \text{ord}_{C_a} X_{\tau j} = \text{minimum}_{1 \leq j \leq 3} \text{ord}_{C_a} X_j$ by Proposition (6.1). In particular, if X is an element of $H^0(M_t, \theta_W|_{M_t})$, we have $\text{ord}_{C_i} X \geq -1$ for each i . Similarly let $\omega = \sum_{j=1}^3 Y_j \tilde{d}y_j$ be a rational 1-form such that $Y_j \in A$. We

define $\text{ord}_{\hat{C}_i} \omega$ and $\text{ord}_{C_i} \omega$ in the same way. Then we have

Lemma (7.4). (i) Let X be as above and assume that $\{X_j\}$ ($j = 1, 2, 3$) have regular forms on C_a and assume that $\text{ord}_{C_a} X \leq -2$ for some a . Then X is not a holomorphic section of θ_W over M_t .

(ii) Let $D = \sum_{i=0}^3 n_i C_i + D'$ be a divisor on M_t such that the support of D' does not include any of C_i ($i=0, \dots, 3$). Let ω be as above. Assume that $\{Y_j\}$ ($j=1, 2, 3$) have regular forms on C_a for some a . If $\text{ord}_{C_a} \omega \leq -n_a$, the restriction of ω to M_t is not contained in $H^0(M_t, \Omega_W^1|_{M_t}(D))$.

For the rest of the section, we consider the following example. Let

$$f_{\Delta}(z) = z_0^2 z_1 z_2^4 + z_1^2 z_2 z_3^4 + z_2^2 z_3 z_0^4 + z_3^2 z_0 z_1^4$$

and let $f(z) = f_{\Delta}(z) + \sum_{i=0}^3 z_i^{11}$. Let $P = {}^t(1, 1, 1, 1)$. As $\Gamma^*(f)$ is invariant under the canonical $\mathbb{Z}/4\mathbb{Z}$ -action, we can take Σ^* to be $\mathbb{Z}/4\mathbb{Z}$ -invariant and Σ^* is canonical in the sense of [12]. Namely we have $P_0 = {}^t(1, 2, 3, 1)$, $P_1 = {}^t(1, 1, 2, 3)$, $P_2 = {}^t(3, 1, 1, 2)$ and $P_3 = {}^t(2, 3, 1, 1)$. Let $\sigma = (P, P_0, P_1, R)$ where $R = (P_2 + 2P_0 + 3P_1 + 2P) / 5 = {}^t(2, 2, 3, 3)$. Let $M = E(P)$. The defining equation of M in C_{σ}^3 is

$$h(y) = y_1^5 y_3^2 + y_2^5 y_3^3 + y_3 + 1 = 0.$$

We have shown in Example (9.11) of [12] that $\pi_1(M) = \mathbb{Z}/5\mathbb{Z}$ and $q = p_g = 0$. This surface is known as a Godeaux surface. As l is 11, the dimension of the embedded deformation is 8. The corresponding embedded monomials are: $y_2 y_3$, $y_2^3 y_3^2$, $y_1 y_3$, $y_1 y_2 y_3$, $y_1 y_2^2 y_3^2$, $y_1^2 y_3$, $y_1^2 y_2^2 y_3^2$ and $y_1^3 y_2 y_3^2$. See [11]. Let $h(y, t)$ be as before. As numerical data, we have $K \sim 2C_3 - C_2 \sim 2C_1 - C_0$ and $C_i^2 = 1$ and $K^2 = 1$. Here K is a canonical divisor. By the Riemann-Roch theorem, we have $\chi(\theta_t) = -8$. We will show that

Theorem (7.5). We have $H^0(M_t, \theta_t) = H^2(M_t, \theta_t) = 0$, $H^1(M_t, \theta_t) \cong \mathbb{C}^8$ and the Kodaira-Spencer map

$$\delta \cdot \xi^e : T_t U^e \rightarrow H^1(M_t, \theta_t)$$

is an isomorphism.

Compare with the construction of the moduli space of the Godeaux surfaces by Miyaoka [8]. Note that $\mathbb{Z}/4\mathbb{Z}$ acts canonically on U^e so that $M_t \cong M_{gt}$ for $g \in \mathbb{Z}/4\mathbb{Z}$.

Lemma (7.6). $H^0(M_t, \theta_W|_{M_t}) \cong \mathbb{C}^3$ and $H^2(M_t, \theta_W|_{M_t}) = 0$.

Proof. Let $\tau = (P, P_2, P_3, R')$ where $R' = {}^t(3, 3, 2, 2)$. We denote $y_{\tau i}$ by u_i for simplicity. Then we have $y_1 = u_1^{-2} u_2$, $y_2 = u_1^{-3} u_2^2$ and $y_3 = u_1^5 u_2^{-5} u_3^{-1}$. Let $X \in H^0(M_t, \theta_W|_{M_t})$. By the GAGA-principle, X can be expressed in $\mathbb{C}_\sigma^3 \cap M_t$ as $\sum_{j=1}^3 X_j \frac{\tilde{\partial}}{\partial y_j}$ where $X_j \in A$.

Assertion. We can assume that X_j has a regular form

on C_2 and C_3 simultaneously.

Proof. We may first assume that $\text{ord}_{\hat{C}_3} X_i = \text{ord}_{C_3} X_i$, using the irreducibility of h^3 in A . Assume that X_i has not a regular form on C_2 . We substitute $h^2(y)y^\nu$ by $(h(y,t) - h^2(y,t))y^\nu$ to change X_i in a regular form on C_2 in a finite steps. Note that this operation does not decrease $\text{ord}_{\hat{C}_3} X_i$. Thus if we change X_i in a regular form X'_i on C_2 , we have

$$\text{ord}_{C_3} X_i = \text{ord}_{C_3} X'_i \geq \text{ord}_{\hat{C}_3} X'_i \geq \text{ord}_{\hat{C}_3} X_i.$$

This implies that $\text{ord}_{\hat{C}_3} X'_i = \text{ord}_{C_3} X'_i$ by the regularity assumption on C_3 . Assume that the monomial y^ν has a non-zero coefficient in X_i . As we have

$$y^\nu = u_1^{-2\nu_1-3\nu_2+5\nu_3} u_2^{\nu_1+2\nu_2-5\nu_3} u_3^{-\nu_3},$$

we must have $\nu_1 + 2\nu_2 + 1 \geq 5\nu_3 \geq 2\nu_1 + 3\nu_2 - 1$. Combine this with $\nu_1 \geq -\delta_{i1}$, $\nu_2 \geq -\delta_{i2}$ where δ_{ij} is the Kronecker's symbol. The possible cases are $y_2^2 y_3 \frac{\tilde{\partial}}{\partial y_i}$ ($i=1,2,3$), $y_1^2 y_2^{-1} \frac{\tilde{\partial}}{\partial y_2}$, $y_1 y_2^{-1} \frac{\tilde{\partial}}{\partial y_2}$, $y_1^{-1} y_2 \frac{\tilde{\partial}}{\partial y_1}$ and $\frac{\tilde{\partial}}{\partial y_i}$. After checking their linear combinations in detail, we conclude that $H^0(M_t, \theta_W | M_t) = \text{Can}(M_t, \theta_W)$.

Now we consider $H^2(M_t, \theta_W | M_t)$. By the Serre duality, this is isomorphic to $H^0(M_t, \Omega_W^1(K)) \cong H^0(M_t, \Omega_W^1 | M_t(2C_1 - C_0))$

where Ω_W^1 is the sheaf of the germs of 1-forms on W . Let $\omega = \sum_{i=1}^3 Y_i \tilde{d}y_i$ be a rational 1-form with $Y_i \in A$ and assume that the restriction of ω is in $H^0(M_t, \Omega_W^1|_{M_t}(2C_1 - C_0))$. Let y^v be a monomial with a non-zero coefficient in Y_i . Then by Lemma (7.4), we have $\nu_1 \geq -2 + \delta_{i1}$, $\nu_2 \geq 1 + \delta_{i2}$ and $\nu_1 + 2\nu_2 \geq 5\nu_3 \geq 2\nu_1 + 3\nu_2$. This has no integral solution. This implies that $H^2(M_t, \theta_W|_{M_t}) = 0$, completing the proof of Lemma (7.6).

Proof of Theorem (7.5). We consider the exact sequence (1.4). Considering the section φ of $H^0(M_t, \nu_t)$ such that $\varphi_\sigma = 1$, we see that $N_t = [5C_3]$. Thus by Riemann-Roch theorem, we have $\chi(\nu_t) = 11$, $\chi(\theta_t) = -8$ and $\chi(\theta_W|_{M_t}) = 3$. This implies that $H^1(M_t, \theta_W|_{M_t}) = H^2(M_t, \nu_t) = 0$ and $H^2(M_t, \theta_t) = H^0(M_t, \theta_t) = 0$ and $H^1(M_t, \theta_t) \cong \mathbb{C}^8$. This completes the proof by Theorem (6.8).

References

- [1] V.I. Danilov, The geometry of toric varieties, Russian Math. Surveys, 33:2 (1978), 97-154.
- [2] M. Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona, Ann. Sci. Ecole Norm. Sup., (4) 3 (1970), 507-588.

- [3] F. Ehlers, Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einer isolierter Singularitäten, Math. Ann., 218 (1975), 127-156.
- [4] N. Hashimoto and M. Oka, Example of an algebraic surface whose effective deformation is not injective, in preparation.
- [5] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, Toroidal Embeddings, 339, Springer, Berlin-Heidelberg-New York, 1973.
- [6] K. Kodaira and D.C. Spencer, On deformations of complex structures I,II, Annals of Math., 67 (1958), 328-466.
- [7] K. Kodaira, Complex Manifolds and Deformation of Complex Structures, Springer, Berlin-Heidelberg-New York, 1985.
- [8] Y. Miyaoka, Tricanonical Maps of Numerical Godeaux Surfaces, Inventiones Math., 34 (1976), 99-111.
- [9] T. Oda, Lectures on torus embeddings and applications, 58, Springer-Verlag, Berlin-Heidelberg-New York, 1978.
- [10] T. Oda, Convex body and algebraic geometry (in Japanese), Kinokuniya, Tokyo, 1985.
- [11] M. Oka, Examples of Algebraic Surfaces with $q = 0$ and $p_g \leq 1$ which are Locally Hypersurfaces, preprint.

[12] M. Oka, On the Resolution of Hypersurface Singularities, Advanced Study in Pure Mathematics, 8 (1986), 405-436.

[13] J.P. Serre, Géométrie Algébrique et Géométrie Analytique, Ann. Inst. Fourier, 36 (1956), 1-42.